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DETERMINATION OF THE CRITICAL TORQUE INDUCING  
BUCKLING IN A TWISTED SPHERICAL SHELL  
SUBJECT TO INTERNAL OR EXTERNAL PRESSURE

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PROBLEM: The problem is illustrated by Figs. 2, 3 and 4. A spherical shell is twisted by two equilibrating torques applied by means of flanges at the poles of the sphere. The problem consists in determining the critical value (buckling value) of the torques and in analyzing the shape of the buckling impression (furrow) appearing near the flanges as buckling deformation. Internal pressure  $p$  may be present in the shell.

TOTAL BUCKLING VS. LOCAL BUCKLING: In case of a straight rod buckling under compression we have an example of total buckling. Every inch of the rod is deformed into a curved shape. There is no region of conspicuous concentration of deformation as against a region of relative absence of deformation. Every part, big or small, of the rod is engaged in deformation. In this sense, we call it a case of total buckling.

In case of spherical shells we may observe an entirely different picture: that of local buckling. By applying a negligibly small effort, a very thin spherical brass shell can be pushed in locally forming a combination of 2 spherical caps: a big cap of the undisturbed original sphere and a small cap in buckled position (Fig.1).

THE EXPERIMENT: LOCAL CHARACTER OF BUCKLING IN TORSION: Experiments made at the Watervliet Arsenal Laboratories have shown that the buckling of a spherical shell in torsion begins as local buckling concentrated close by the torque-transmitting flange of the torque producing machine. The shape of the buckled area is oblong-elliptic, stretching in an oblique direction from the edge of the flange. The buckled area resembles a fold or rather a furrow. Fig. 2 shows in a drawing the initial stage of buckling. Figs. 3 and 4 are photographs of more advanced stages in which the furrow is conspicuous enough to be seen and recognized in a photographic reproduction.

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OUTLINE OF BUCKLING ANALYSIS BY THE ENERGY METHOD: Buckling equations are essentially energy balance equations. They state that the strain energy accumulated prior to the onset of buckling has reached a level sufficient to do the work necessary to produce the buckled shape. The external forces and moments pitch in and help in the performance of that work. The buckling equation of our problem will consist of an accurate balance of strain energy available for release through buckling and the work done by the external efforts on one side, and the energy required for inflection in the critical area to make it buckle, on the other side.

Each item mentioned above can be computed analytically. The strain energy of a shell consists of 2 parts:  $V_1$ , the strain energy due to the stretching of the middle surface, and  $V_2$ , the strain energy due to bending of the middle surface. The general principles for the computation of  $V_1$  and  $V_2$  have been established by A.E.H. Love and introduced to engineering analysis by S. P. Timoshenko. We will use methods of classical differential geometry on that occasion. The experiment shown in Fig. 8 will lead us to a reasonable assumption concerning the character of the local buckling process. From there on, we will arrive at the results by purely mathematical processes, making no further assumptions to suit the situation.

DIFFERENTIAL GEOMETRY OF THE BUCKLED SPHERICAL SHELL: we will use polar (spherical) coordinates  $r$ ,  $\theta$ ,  $\phi$  defined by

$$x = r \bar{p} \cos \phi$$

$$y = r \bar{p} \sin \phi$$

$$z = r p$$

in which

$$p = \cos \theta, \quad \bar{p} = \sin \theta$$

Let  $a$  be the radius of the middle surface of the shell. A point  $P$  on the middle surface has polar coordinates

$$a, \theta, \phi.$$

Let the polar components of the displacement vector of  $P$  be

$$u_r = a \alpha$$

$$u_\theta = a \beta$$

$$u_\phi = a \gamma$$

SESSION B-III-3

TITLE: Determination of the Critical Torque Inducing Buckling in  
a Twisted Spherical Shell Subject to Internal or External  
Pressure

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ABSTRACT: The problem of determining the critical value of the torque  
T inducing buckling in a pressurized twisted spherical shell has been  
solved under assumption of local non-analytic buckling. The critical  
relation between the values of the torque T and pressure p has been  
explicitly determined.

SESSION B-III-4

in which

$$\alpha = \alpha(\theta, \phi), \quad \beta = \beta(\theta, \phi), \quad \gamma = \gamma(\theta, \phi)$$

are dimensionless functions of  $\theta$  and  $\phi$ . We will denote their derivatives with respect to  $\theta$  and  $\phi$  by subscripts, such as in

$$\frac{\partial \alpha}{\partial \theta} = \alpha_{\theta}, \quad \frac{\partial^2 \beta}{\partial \theta \partial \phi} = \beta_{\theta \phi}, \quad \dots$$

**THE BASIC STRESS FIELD:** The stress field in the twisted shell at the critical value of the torque  $T$  prior to buckling is called the basic stress field. Its knowledge is important because in buckling a part of its strain energy is released for the formation of the buckled shape. The determination of the basic field, meaning the determination of all displacements and stresses in a spherical shell of radius  $a$  and thickness  $b$  under the action of the torque  $T_0$  is a problem of 3-dimensional mathematical theory of elasticity. The solution, in polar coordinates  $r, \theta, \phi$  reads:

$$u_{\phi} = \frac{T_0 r}{4\pi a^2 h G} \left[ \frac{\bar{p}}{2} \log \frac{1+p}{1-p} + \frac{p}{\bar{p}} \right]$$

$$\tau_{\theta \phi} = -\frac{T_0}{2\pi a^2 h} \cdot \frac{1}{\bar{p}^2}$$

$$u_r = u_{\theta} = \epsilon_r = \epsilon_{\theta} = \epsilon_{\phi} = \tau_{r\theta} = \tau_{r\phi} = 0$$

We note that the only polar stress component that does not vanish is the shearing stress  $\tau_{\theta \phi}$ .

THE BUCKLING EQUATION: The buckling equation is an energy balance stating that the strain energy of the buckling deformation comes from two sources: from work done by external forces and from an influx of strain energy of the basic stress field. It can be found by systematically applying the mathematical theory of surfaces and space curves (differential geometry) to the occasion and it reads in the case when there is no internal pressure in the shell as follows:

$$\begin{aligned}
& \frac{T_0}{2\pi a^2} \iint \frac{1}{\bar{\rho}^4} \left[ (\alpha_\theta - \beta)(\alpha_\phi - \gamma \bar{\rho}) \bar{\rho} + \right. \\
& \left. + (\gamma \bar{\rho} - \beta_\phi)(\beta \bar{\rho} + \gamma_\phi + \alpha \bar{\rho}) - \gamma_\theta (\alpha + \beta_\theta) \bar{\rho}^2 \right] dA = \\
& = \frac{D}{4a^2} \iint \left[ (1+\nu) \left( 2\alpha + \alpha_{\theta\theta} + \frac{\alpha_\theta \bar{\rho}}{\bar{\rho}} + \frac{\alpha_{\phi\phi}}{\bar{\rho}^2} \right)^2 + \right. \\
& \left. + (1-\nu) \left( \frac{\alpha_\theta \bar{\rho}}{\bar{\rho}} - \alpha_{\theta\theta} + \frac{\alpha_{\phi\phi}}{\bar{\rho}^2} \right)^2 + 4(1-\nu) \left( \frac{\alpha_\phi \bar{\rho}}{\bar{\rho}^2} - \frac{\alpha_{\theta\phi}}{\bar{\rho}} \right)^2 \right] dA + \\
& + \frac{Eh}{2(1-\nu^2)} \iint \left[ \left( 2\alpha + \beta_\theta + \frac{\beta \bar{\rho} + \gamma_\phi}{\bar{\rho}} \right)^2 - \right. \\
& \left. - 2(1-\nu) (\alpha + \beta_\theta) \left( \alpha + \frac{\beta \bar{\rho} + \gamma_\phi}{\bar{\rho}} \right) + \right. \\
& \left. + \frac{1-\nu}{2} \left( \gamma_\theta + \frac{\beta_\phi - \gamma \bar{\rho}}{\bar{\rho}} \right)^2 \right] dA
\end{aligned}$$



CHARACTERISTIC ASSUMPTIONS: The problem of solving the buckling equation consists of determining 3 functions  $\alpha(\theta, \phi)$ ,  $\beta(\theta, \phi)$ ,  $\gamma(\theta, \phi)$  compatible with boundary conditions which will minimize the value of the torque  $T_0$ . Inspecting the structure of various integrals appearing we see that the bending energy (term with the coefficient D) depends on  $\alpha$  alone. This points to a prominence of the function  $\alpha(\theta, \phi)$  as compared with  $\beta(\theta, \phi)$  and  $\gamma(\theta, \phi)$ . We now make our first characteristic assumption: we assume

$$\beta(\theta, \phi) \equiv 0, \quad \gamma(\theta, \phi) \equiv 0.$$

We drop from the bending energy integral all terms in  $\alpha, \alpha_\theta$  and  $\alpha_\phi$ . This is the second characteristic assumption.

GEOMETRY OF THE BUCKLING FURROW ON A SPHERICAL SHELL: The basic guidance for judgment is given by the photographs Figs. 3 and 4 showing experiments made at the Watervliet Arsenal Laboratories with spherical shells buckling in torsion. Buckling begins as a localized effect concentrated close by the torque-transmitting flange of the torsion test machine. The shape of the buckled area is oblong-elliptic, stretching in an oblique direction from the edge of the flange. The buckled shape resembles a fold or rather a furrow. At the beginning, one single furrow appears. Photographic reproductions of that initial stage were unsatisfactory and have not been included in the present report. At a later stage of buckling, one or two more furrows of identical appearance would show up. The depth of the furrows at that stage was sufficient to produce illumination effects giving good photographic reproductions. The photographs included here show a stage at which 2 furrows have been formed. The theoretical analysis refers to the initial stage of the formation of the first furrow.

The buckling equation gives

$$T_0 = 8.583 \frac{E h^2 r^2}{(1-\nu) \sqrt{1+\nu} a}$$

in which  $r$  is the radius of the flange bonded to the shell. For the minor semiaxis of the elliptic furrow we have

$$b = \frac{1.354}{\sqrt[4]{1+\nu}} \sqrt{a h}$$

THE COMPLETE BUCKLING DIAGRAM IN PRESENCE OF INTERNAL PRESSURE  $p$ :  
 Let  $T_0$  be the buckling torque in absence of pressure, and  $p_0$  the external buckling pressure in absence of a torque. The value of  $p_0$  resultant from the present computation is

$$p_0 = \frac{(\sqrt{6} + 3\sqrt{2})}{3(1-\nu)\sqrt{1+\nu}} \cdot \frac{E h^2}{a^2}$$

The complete buckling diagram is shown in Fig. 5. It extends over the ranges of dimensionless variables

$$-1 \leq \frac{p}{p_0} < \infty, \quad 0 \leq \frac{T}{T_0} < \infty$$

in the range

$$-\sqrt{\frac{2}{3}} \leq \frac{p}{p_c} < \infty$$

the diagram is a straight line given by

$$\frac{T}{T_0} = 1 + \sqrt{\frac{2}{3}} \cdot \frac{p}{p_0}$$

in the range

$$-1 \leq \frac{p}{p_0} \leq -\sqrt{\frac{2}{3}}$$

the diagram is part of the ellipse

$$\frac{T}{T_0} = \frac{1}{\sqrt{3}} \cdot \sqrt{1 - \frac{p^2}{p_c^2}}$$

According to this result, it is possible to make a membrane shell torque resistant by inflating it with internal pressure. The rectilinear part of the diagram in Fig. 5 becomes

$$T = \pi a r^2 p \quad p \geq 0$$

for the membrane shell.

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CONCLUSIONS: Based on exact methods of differential geometry and on accepted engineering approximations in applying those methods to shell theory, it has been possible to successfully deal with a pressurized spherical shell buckling under action of twisting torques. The critical relation between the values of the torque and pressure under whose simultaneous action buckling takes place has been obtained. Inflation of the shell by internal pressure increases the resistance against torsional buckling.

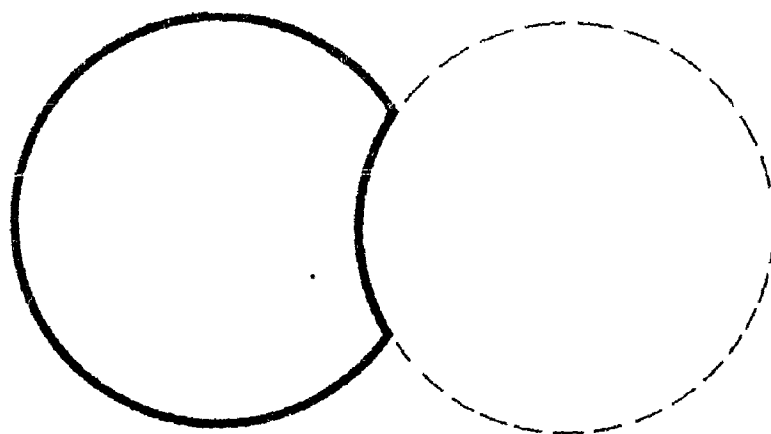


Figure 1. Local buckling of a spherical shell by formation of a reflected cap.

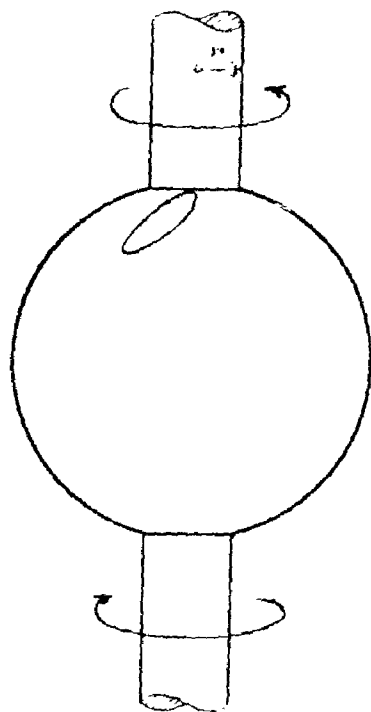


Figure 2. Buckling of a spherical shell under twisting torques.

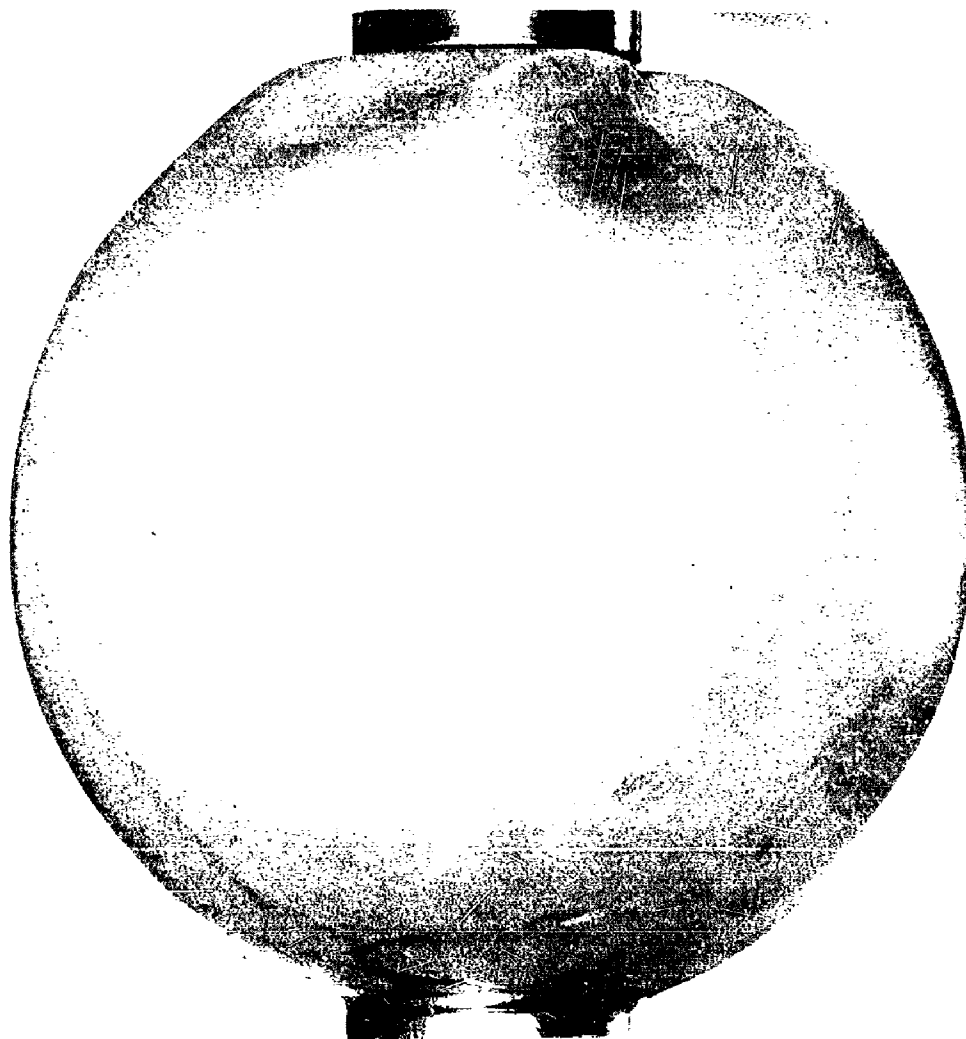


Figure 3. Actual test photograph of a spherical shell in local buckling under twisting torques.

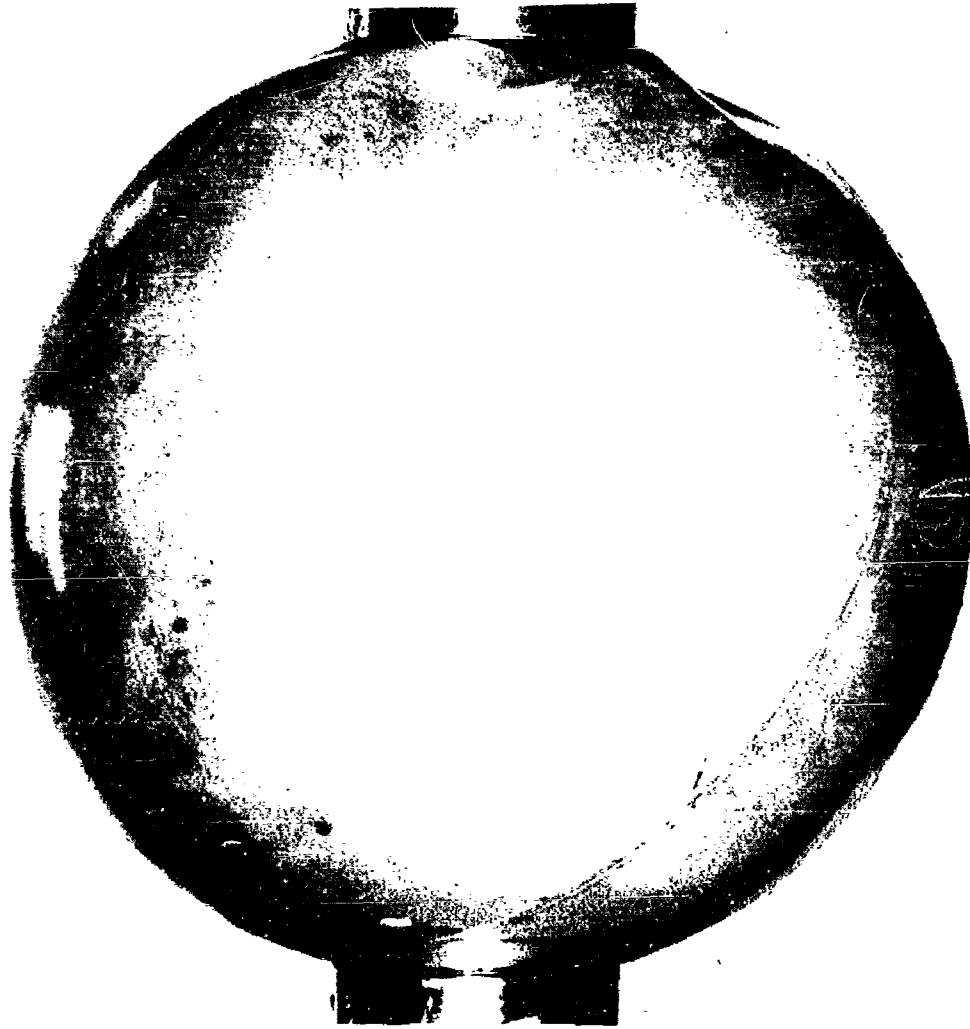


Figure 4. Actual test photograph of a spherical shell in local buckling under twisting torques.

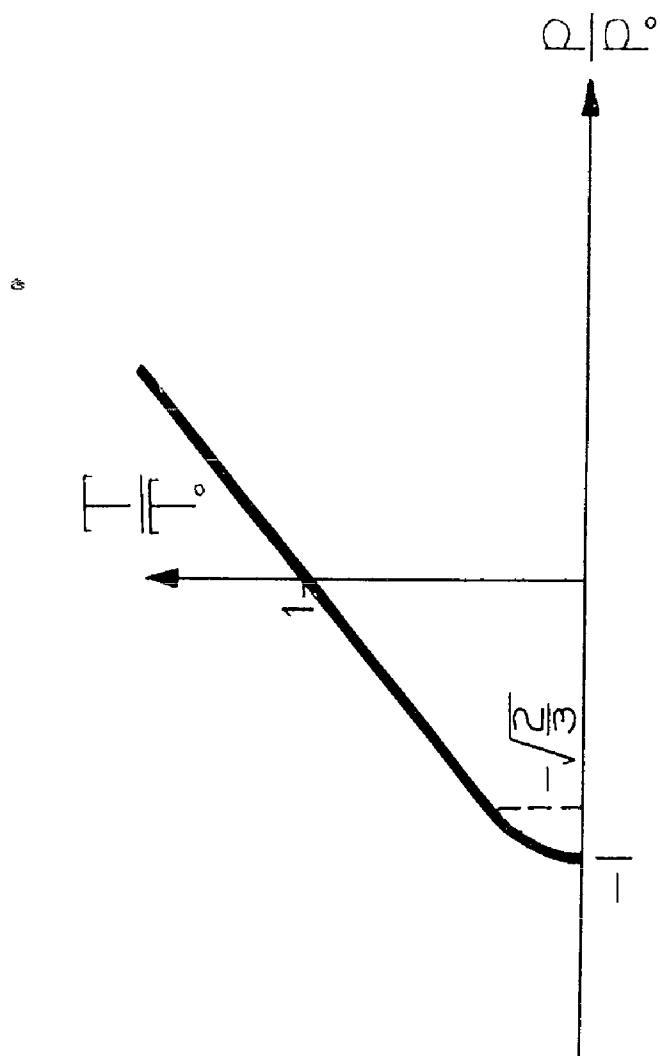


Figure 5. The complete critical diagram

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